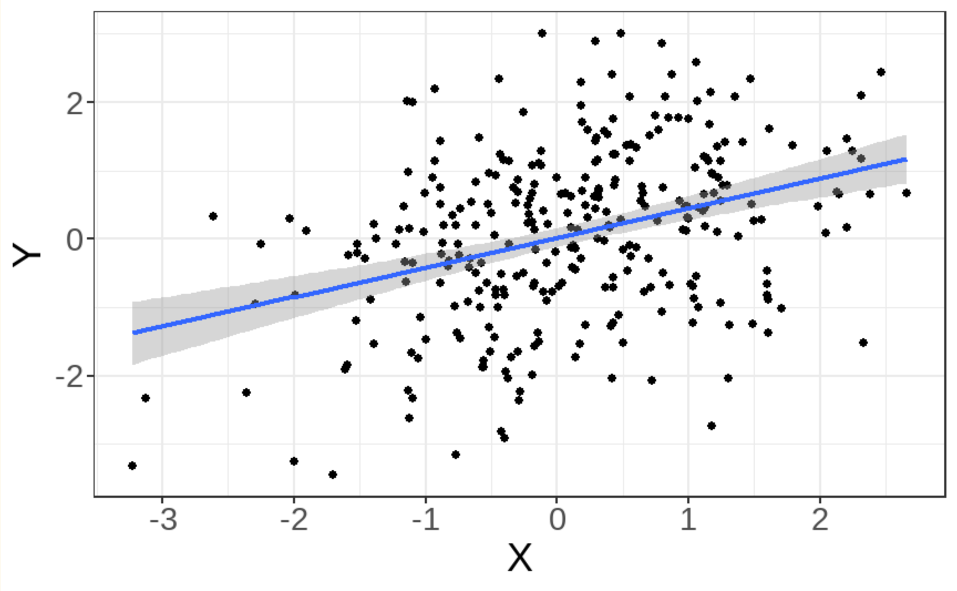
Multiple and logistic regression

2021/04/13

Linear regression - a (brief) recap

Linear regression

```
## `geom_smooth()` using formula 'y ~ x'
```



Our job is to figure out the mathematical relationship between our *predictor(s)* and our *outcome*.

$$Y = b_0 + b_1 X_i + \varepsilon_i$$

Linear regression

$$Y = b_0 + b_1 X_i + \varepsilon_i$$

Linear regression

$$Y = b_0 + b_1 X_i + \varepsilon_i$$

Y - The outcome - the dependent variable.

b_0 - The *intercept*. This is the value of Y when X = 0.

b_1 - The regression coefficients. This describes the steepness of the relationship between the outcome and *slope(s)*.

X_i - The predictors - our independent variables.

ε_i - The *random error* - variability in our dependent variable that is not explained by our predictors.

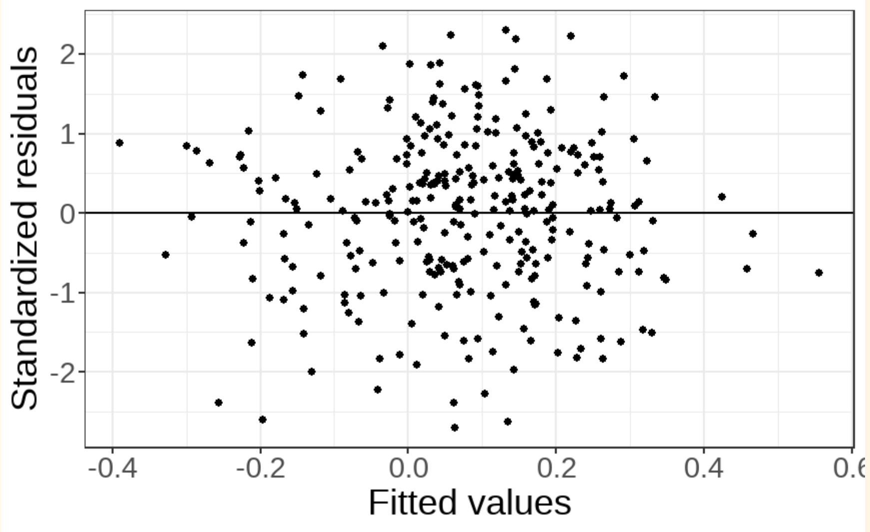
Regression assumptions

Regression assumptions

Linear regression has a number of assumptions:

- Normally distributed errors
- Homoscedasticity (of errors)
- Independence of errors
- Linearity
- No perfect multicollinearity

Normally distributed errors

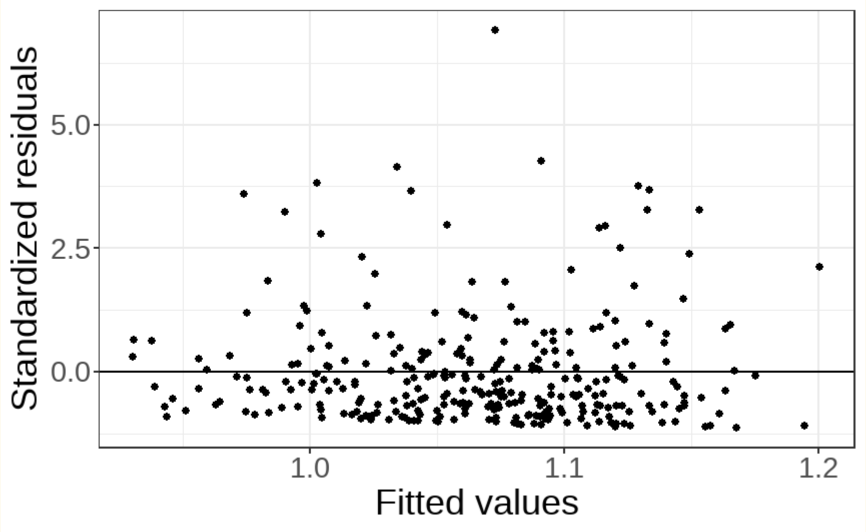


The *errors*, ε_i, are the variance left over after your model is fit.

An example like that on the left is what you want to see!

There is no clear pattern; the dots are evenly distributed around zero.

Skewed errors



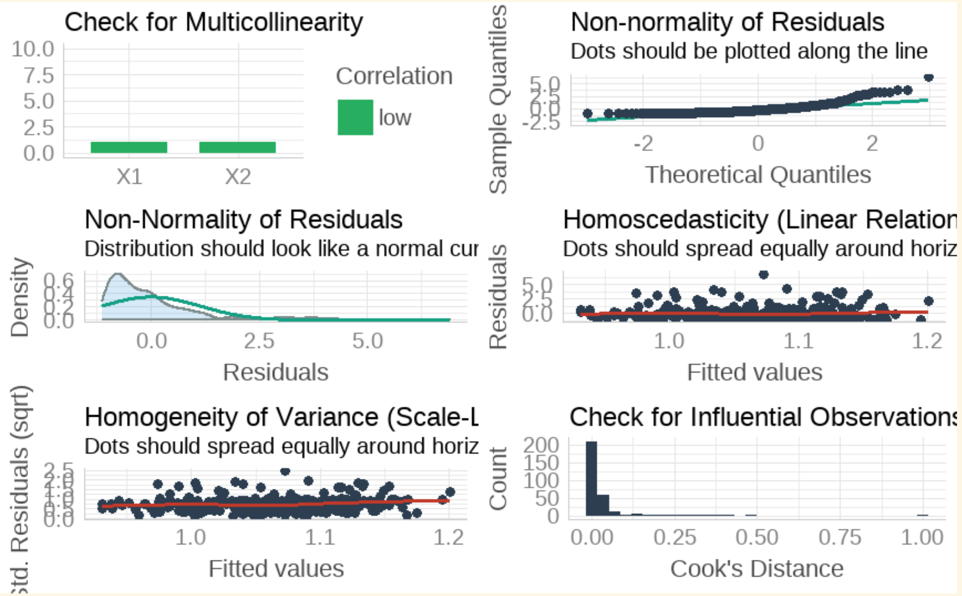
In contrast, the residuals on the left are skewed.

This most often happens with data that are *bounded*. For example, *reaction times* cannot be below zero; negative values are impossible.

Checking assumptions

The `performance` package has a very handy function called `check_model()`, which shows a variety of ways of checking the assumptions all at once.

```
library(performance)
check_model(test_skew)
```

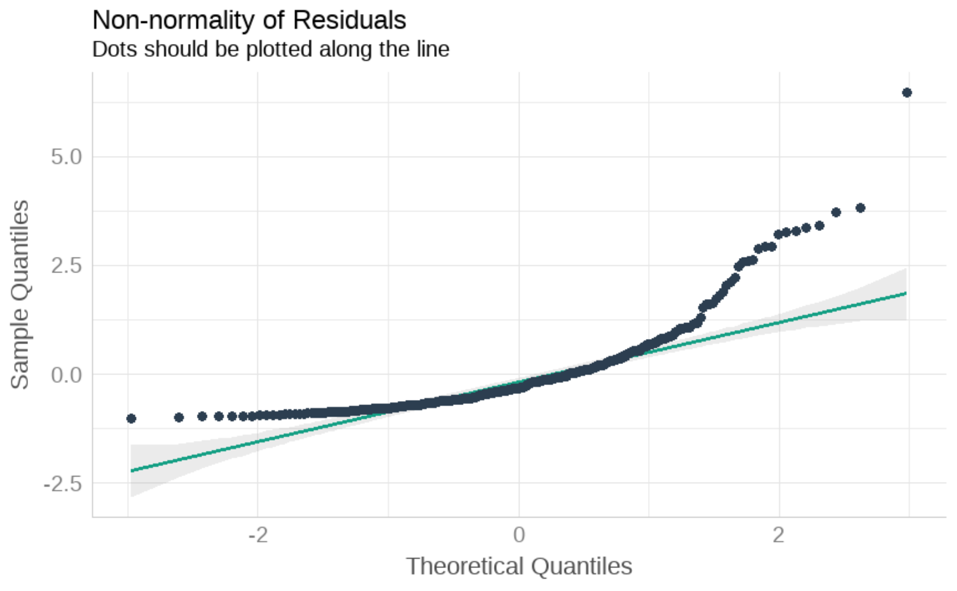


Checking assumptions

You can follow up suspicious looking plots with individual functions like `check_normality()`, which uses `shapiro.test()` to check the residuals and also provides nice plots.

Rely on the plots more than significance tests...

```
plot(check_normality(test_skew),
     type = "qq")
```




```
## Warning: Non-normality of residuals detected (p < .001).
```

So, about violated assumptions? 😱

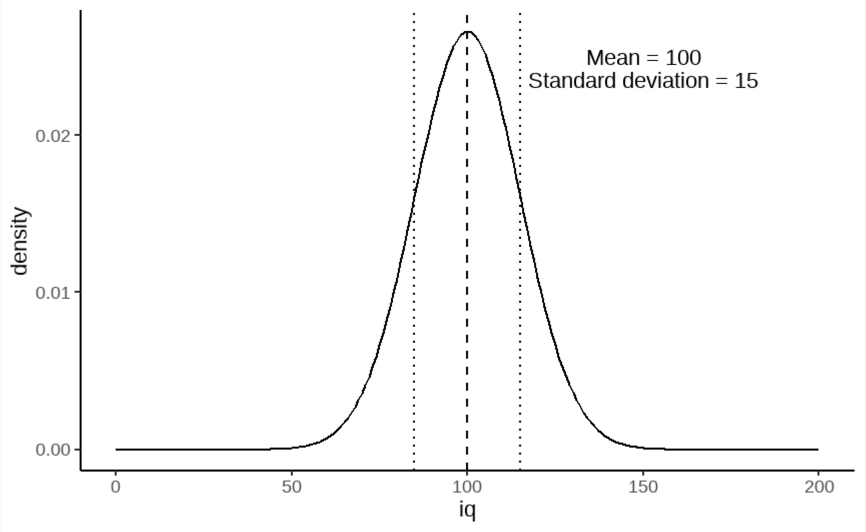
1) We can think about transformations 😱

2) We could consider *non-parametric stats* - things like `wilcox.test()`, `friedman.test()`, `kruskal.test()`, all of which are based on rank transformations and thus are really more like point 1 😱

3) We should think about **why** the assumptions might be violated. Is this just part of how the data is *generated*? 🤔

Generalized Linear Models

Distributional families



The *normal* distribution can also be called the *Gaussian* distribution.

The linear regression models we've used so far assume a *Gaussian* distribution of errors.

Generalized linear models

A *Generalized Linear Model* - fit with `glm()` - allows you to specify what type of family of probability distributions the data are drawn from.

The data | With lm | With glm

```
skewed_var <- rgamma(300, 1)
hist(skewed_var)
```

Generalized linear models

A *Generalized Linear Model* - fit with `glm()` - allows you to specify what type of family of probability distributions the data are drawn from.

The data | With lm | With glm

```
lm(skewed_var ~ X1 + X2)
```

```
##
## Call:
## lm(formula = skewed_var ~ X1 + X2)
##
## Coefficients:
## (Intercept)           X1           X2
##     0.99775     -0.04337      0.04234
```

Generalized linear models

A *Generalized Linear Model* - fit with `glm()` - allows you to specify what type of family of probability distributions the data are drawn from.

The data | With lm | **With glm**

```
glm(skewed_var ~ X1 + X2, family = "gaussian")
```

```
##
## Call:  glm(formula = skewed_var ~ X1 + X2, family = "gaussian")
##
## Coefficients:
## (Intercept)           X1           X2
##     0.99775     -0.04337      0.04234
##
## Degrees of Freedom: 299 Total (i.e. Null);  297 Residual
## Null Deviance:        268.9
## Residual Deviance: 267.8     AIC: 825.3
```

Categorical outcome variables

Suppose you have a *discrete*, *categorical* outcome.

Examples of categorical outcomes:

- correct or incorrect answer
- person commits an offence or does not

Examples of counts:

- Number of items successfully recalled
- Number of offences committed

The binomial distribution

A coin only has two sides: heads or tails.

Assuming the coin is fair, the probability - P - that it lands on *heads* is *.5*. So the probability it lands on *tails* - $1 - P$ is also *.5*.

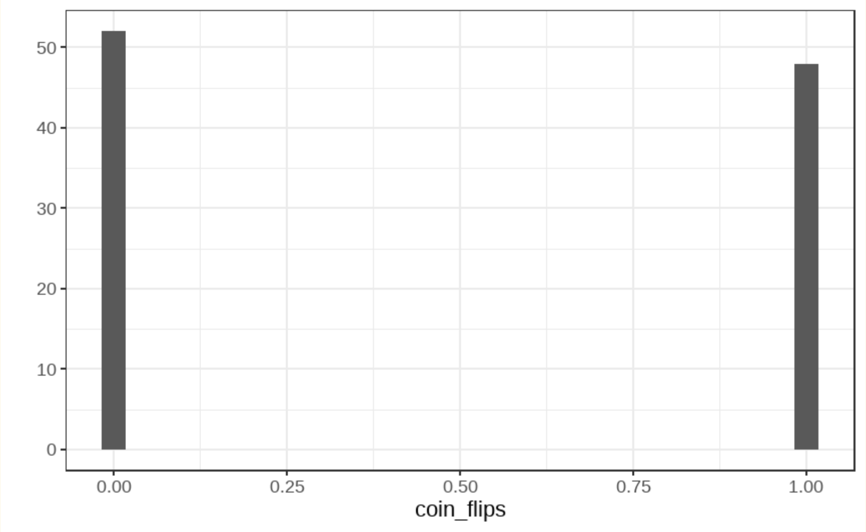
This type of variable is called a **Bernoulli random variable**.

If you toss the coin many times, the count of how many heads and how many tails occur is called a **binomial distribution**.

Binomial distribution

If we throw the coin 100 times, how many times do we get tails?

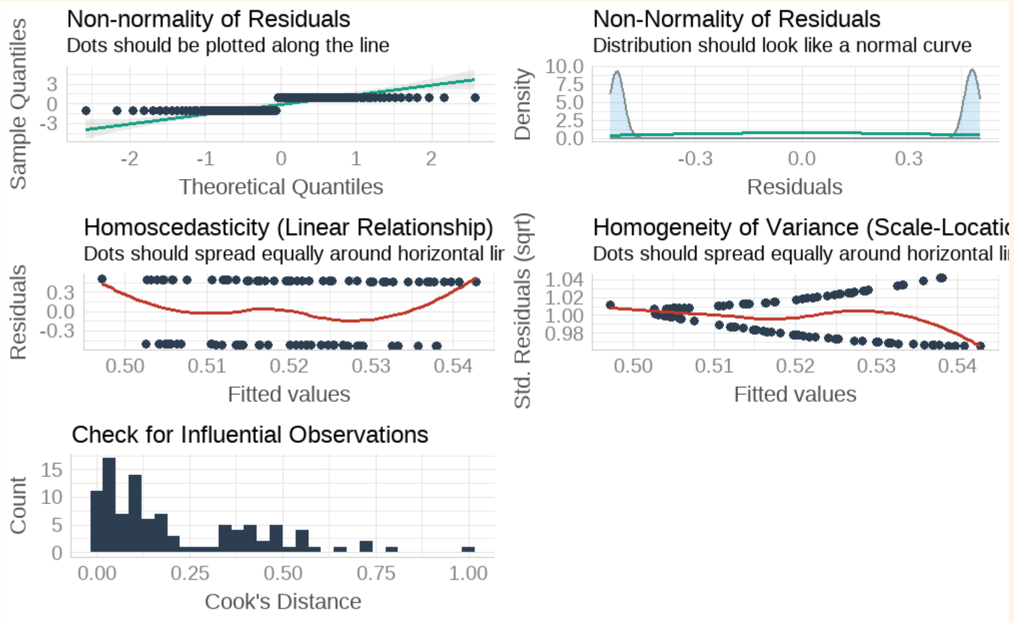
```
coin_flips <- rbinom(n = 100, size = 1, prob = 0.5)
qplot(coin_flips)
```



Binomial distribution

What happens if we try to model individual coin flips with `lm()`?

```
coin_flips <- rbinom(n = 100, size = 1, prob = 0.5)
x3 <- rnorm(100) # this is just a random variable for the purposes of demonstration!
check_model(lm(coin_flips ~ x3))
```



Logistic regression

We get `glm()` to model a `binomial` distribution by specifying the *binomial* family.

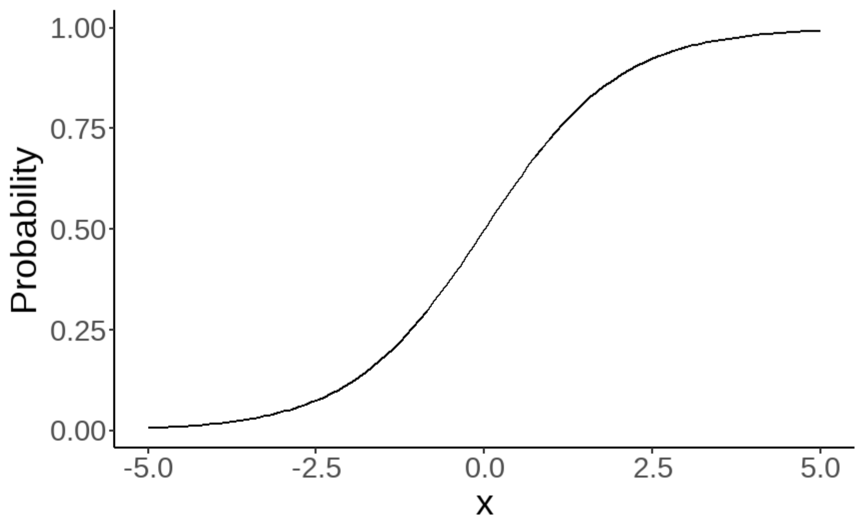
```
coin_flips <- rbinom(n = 100, size = 1, prob = 0.5)
glm(coin_flips ~ 1,
    family = binomial(link = "logit"))
```

```
##
## Call:  glm(formula = coin_flips ~ 1, family = binomial(link = "logit"))
##
## Coefficients:
## (Intercept)
##    -0.04001
##
## Degrees of Freedom: 99 Total (i.e. Null);  99 Residual
## Null Deviance:        138.6
## Residual Deviance: 138.6     AIC: 140.6
```

Logistic regression

The *logit* transformation is used to *link* our predictors to our discrete outcome variable.

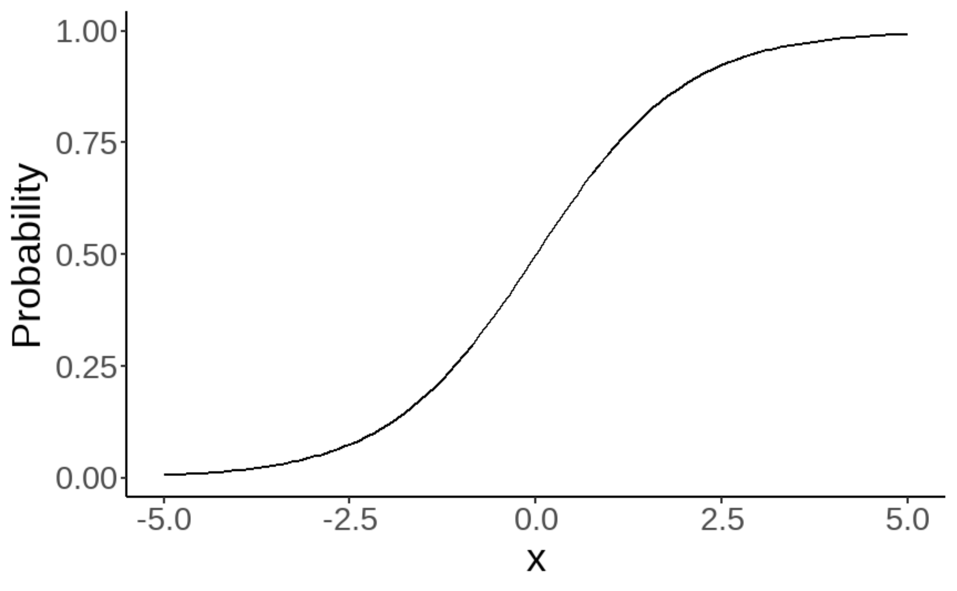
It helps us constrain the influence of our predictors to the range 0-1, and account for the change in *variance* with probability.



Logistic regression

As probabilities approach zero or one, the range of possible values *decreases*.

Thus, the influence of predictors on the *response scale* must also decrease as we reach one or zero.



Logistic regression

$$P(Y) = \frac{1}{1 + e^{-}(b_0 + b_1 X_1 + \varepsilon_i)}$$

$P(Y)$ - The *probability* of the outcome happening.

Logistic regression

$$P(Y) = \frac{1}{1 + e^{-}(b_0 + b_1 X_1 + \varepsilon_i)}$$

$P(Y)$ - The *probability* of the outcome happening.

$\frac{1}{1+e^{-}(...)}$ - The *log-odds* (logits) of the predictors.

Odds ratios and log odds

Odds are the ratio of one outcome versus the others. e.g. The odds of a randomly chosen day being a Friday are 1 to 6 (or 1/6 = .17)

Log odds are the *natural log* of the odds:

$$log(\frac{p}{1-p})$$

The coefficients we get out are *log-odds* - they can be hard to interpret on their own.

```
coef(glm(coin_flips ~ 1, family = binomial(link = "logit")))
```

```
## (Intercept)
## -0.04000533
```

Odds ratios and log odds

If we exponeniate them, we get the *odds ratios* back.

```
exp(coef(glm(coin_flips ~ 1, family = binomial(link = "logit"))))
```

```
## (Intercept)
##   0.9607843
```

So this one is roughly 1:1 heads and tails! But there's a nicer way to figure it out...

Taking penalties

Taking penalties

What's the probability that a particular penalty will be scored?

```
##   PSWQ Anxious Previous         Scored
## 1   18      21       56 Scored Penalty
## 2   17      32       35 Scored Penalty
## 3   16      34       35 Scored Penalty
## 4   14      40       15 Scored Penalty
## 5    5      24       47 Scored Penalty
## 6    1      15       67 Scored Penalty
```

- **PSWQ** = Penn State Worry Questionnaire
- **Anxiety** = State Anxiety
- **Previous** = Number of penalties scored previously

Taking penalties

```
pens <- glm(Scored ~ PSWQ + Anxious + Previous,
            family = binomial(link = "logit"),
            data = penalties)
pens
```

```
##
## Call:  glm(formula = Scored ~ PSWQ + Anxious + Previous, family = binomial(link = "logit"),
##     data = penalties)
##
## Coefficients:
## (Intercept)         PSWQ      Anxious     Previous
##    -11.4926      -0.2514       0.2758       0.2026
##
## Degrees of Freedom: 74 Total (i.e. Null);  71 Residual
## Null Deviance:       103.6
## Residual Deviance: 47.42     AIC: 55.42
```

```
##
## Call:
## glm(formula = Scored ~ PSWQ + Anxious + Previous, family = binomial(link = "logit"),
##     data = penalties)
##
## Deviance Residuals:
##      Min        1Q    Median        3Q       Max
## -2.31374  -0.35996   0.08334   0.53860   1.61380
##
## Coefficients:
##              Estimate Std. Error z value Pr(>|z|)
## (Intercept) -11.49256   11.80175  -0.974  0.33016
## PSWQ         -0.25137    0.08401  -2.992  0.00277 **
## Anxious       0.27585    0.25259   1.092  0.27480
## Previous      0.20261    0.12932   1.567  0.11719
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## (Dispersion parameter for binomial family taken to be 1)
##
##     Null deviance: 103.638  on 74  degrees of freedom
## Residual deviance:  47.416  on 71  degrees of freedom
## AIC: 55.416
##
## Number of Fisher Scoring iterations: 6
```

The response scale and the link scale

The model is fit on the *link* scale.

The coefficients returned by the GLM are in *logits*, or *log-odds*.

```
coef(pens)
```

```
## (Intercept)        PSWQ     Anxious    Previous
## -11.4925608  -0.2513693   0.2758489   0.2026082
```

How do we interpret them?

Converting logits to odds ratios

```
coef(pens)[2:4]
```

```
##        PSWQ    Anxious   Previous
## -0.2513693  0.2758489  0.2026082
```

We can *exponentiate* the log-odds using the **exp()** function.

```
exp(coef(pens)[2:4])
```

```
##      PSWQ   Anxious  Previous
## 0.7777351 1.3176488 1.2245925
```

Odds ratios

An odds ratio greater than 1 means that the odds of an outcome increase.

An odds ratio less than 1 means that the odds of an outcome decrease.

```
exp(coef(pens)[2:4])
```

```
##      PSWQ   Anxious  Previous
## 0.7777351 1.3176488 1.2245925
```

From this table, it looks like the odds of scoring a penalty decrease with increases in PSWQ but increase with increases in State Anxiety or Previous scoring rates.

The response scale

The *response* scale is even *more* intuitive. It makes predictions using the *original* units. For a binomial distribution, that's *probabilities*. We can generate probabilities using the **predict()** function.

```
penalties$prob <- predict(pens, type = "response")
head(penalties)
```

```
##   PSWQ Anxious Previous          Scored      prob
## 1   18      21       56 Scored Penalty 0.7542999
## 2   17      32       35 Scored Penalty 0.5380797
## 3   16      34       35 Scored Penalty 0.7222563
## 4   14      40       15 Scored Penalty 0.2811731
## 5    5      24       47 Scored Penalty 0.9675024
## 6    1      15       67 Scored Penalty 0.9974486
```

Model predictions

Note that these model predictions *don't need to use the original data*. Let's see how the probability of scoring changes as `PSWQ` increases.

Create new data | Make predictions | Plot predictions

```
new_dat <-
  tibble::tibble(PSWQ = seq(0, 30, by = 2),
                 Anxious = mean(penalties$Anxious),
                 Previous = mean(penalties$Previous))
```

Model predictions

Note that these model predictions *don't need to use the original data*. Let's see how the probability of scoring changes as `PSWQ` increases.

Create new data | **Make predictions** | Plot predictions

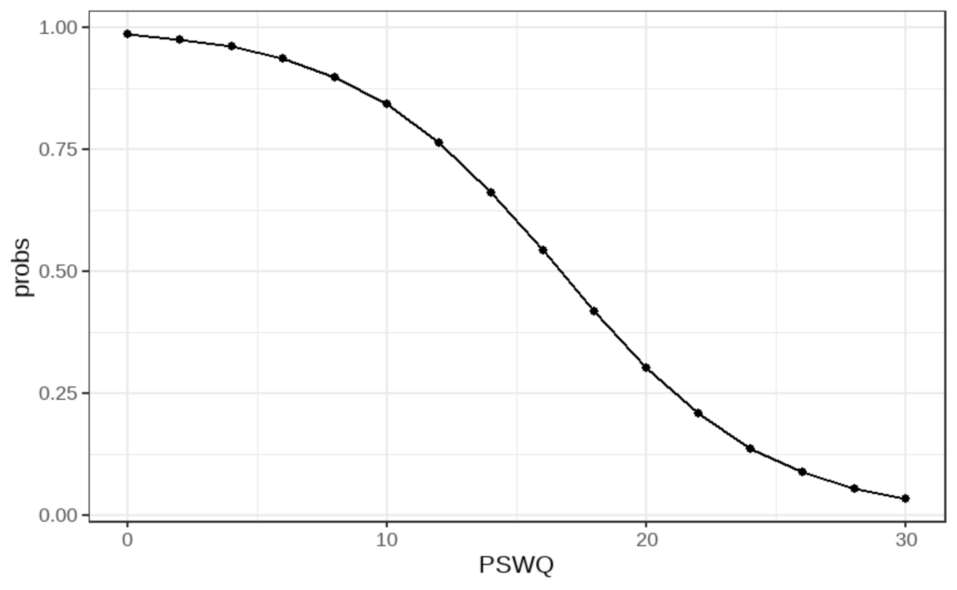
```
new_dat$probs <-
  predict(pens,
          newdata = new_dat,
          type = "response")
```

Model predictions

Note that these model predictions *don't need to use the original data*. Let's see how the probability of scoring changes as `PSWQ` increases.

Create new data | Make predictions | **Plot predictions**

```
ggplot(new_dat, aes(x = PSWQ, y = probs)) +
  geom_point() +
  geom_line()
```



Model predictions

Imagine you wanted to the probability of scoring for somebody with a PSWQ score of 7, an Anxious rating of 12, and a Previous scoring record of 34.

Make the data | Predict log-odds | Predict odds | Predict probabilities

```
new_dat <- tibble::tibble(PSWQ = 7,
                          Anxious = 22,
                          Previous = 34)
```

Model predictions

Imagine you wanted to the probability of scoring for somebody with a PSWQ score of 7, an Anxious rating of 12, and a Previous scoring record of 34.

Make the data | Predict log-odds | Predict odds | Predict probabilities

```
predict(pens, new_dat)
```

```
##          1
## -0.2947909
```

Model predictions

Imagine you wanted to the probability of scoring for somebody with a PSWQ score of 7, an Anxious rating of 12, and a Previous scoring record of 34.

Make the data | Predict log-odds | **Predict odds** | Predict probabilities

```
exp(predict(pens, new_dat))
```

```
##         1
## 0.7446873
```

Model predictions

Imagine you wanted to the probability of scoring for somebody with a PSWQ score of 7, an Anxious rating of 12, and a Previous scoring record of 34.

Make the data | Predict log-odds | Predict odds | Predict probabilities

```
predict(pens, new_dat, type = "response")
```

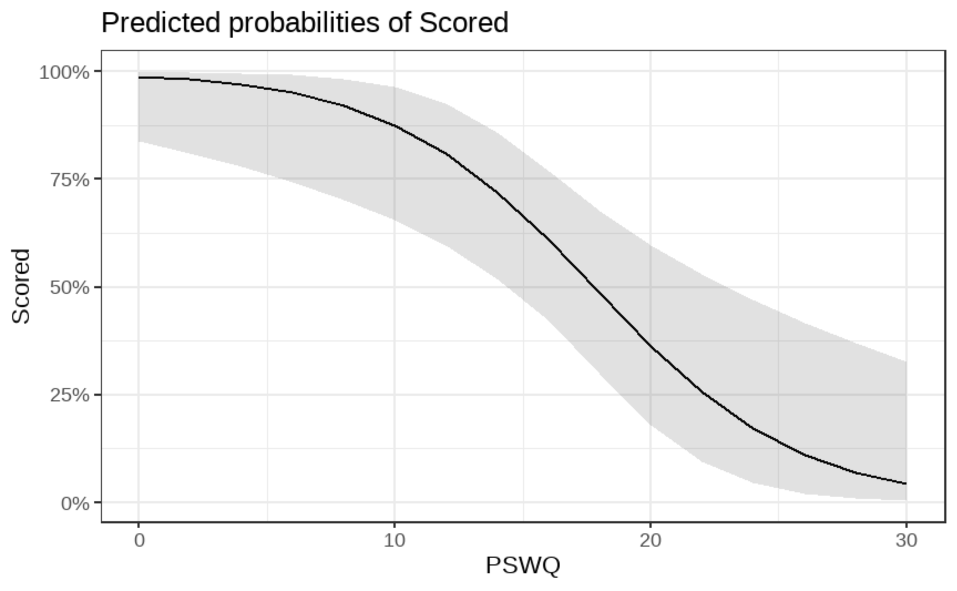
```
##         1
## 0.4268314
```

Plotting

The **sjPlot** package has some excellent built in plotting tools - try the **plot_model()** function.

```
library(sjPlot)
plot_model(pens,
           type = "pred",
           terms = "PSWQ")
```

```
## Data were 'prettified'. Consider using `terms="
```



Results tables

```
sjPlot::tab_model(pens)
```

	Scored		
Predictors	*Odds Ratios*	*CI*	*p*
(Intercept)	0.00	0.00 – 64258.63	0.330
PSWQ	0.78	0.64 – 0.90	**0.003**
Anxious	1.32	0.81 – 2.24	0.275
Previous	1.22	0.96 – 1.61	0.117
Observations	75		
R^2 Tjur	0.594		

The Titanic dataset

The Titanic dataset

The Titanic dataset

```
head(full_titanic)
```

```
## # A tibble: 6 x 12
##   PassengerId Survived Pclass Name    Sex      Age SibSp Parch Ticket  Fare Cabin
##         <dbl>    <dbl>  <dbl> <chr>   <chr>  <dbl> <dbl> <dbl> <chr>  <dbl> <chr>
## 1           1        0      3 Braund~ male      22     1     0 A/5 2~  7.25 <NA>
## 2           2        1      1 Cuming~ fema~     38     1     0 PC 17~ 71.3  C85
## 3           3        1      3 Heikki~ fema~     26     0     0 STON/~  7.92 <NA>
## 4           4        1      1 Futrel~ fema~     35     1     0 113803 53.1  C123
## 5           5        0      3 Allen,~ male      35     0     0 373450  8.05 <NA>
## 6           6        0      3 Moran,~ male      NA     0     0 330877  8.46 <NA>
## # ... with 1 more variable: Embarked <chr>
```

Downloaded from Kaggle

The Titanic dataset

```
VARIABLE DESCRIPTIONS:
survival        Survival
                (0 = No; 1 = Yes)
pclass          Passenger Class
                (1st; 2nd; 3rd)
name            Name
sex             Sex
age             Age
sibsp           N Siblings/Spouses Aboard
parch           N Parents/Children Aboard
ticket          Ticket Number
fare            Passenger Fare
cabin           Cabin
embarked        Port of Embarkation
                (C = Cherbourg;
                 Q = Queenstown;
                 S = Southampton)
```

```
SPECIAL NOTES:
Pclass is a proxy for socio-economic status (SES)
 1st ~ Upper; 2nd ~ Middle; 3rd ~ Lower

Age is in Years; Fractional if Age less than One (1)
 If the Age is Estimated, it is in the form xx.5

With respect to the family relation variables (i.e. sibsp and parch)
some relations were ignored.  The following are the definitions used
for sibsp and parch.

Sibling:  Brother, Sister, Stepbrother, or Stepsister of Passenger
           Aboard Titanic
Spouse:   Husband or Wife of Passenger Aboard Titanic
           (Mistresses and Fiances Ignored)
Parent:   Mother or Father of Passenger Aboard Titanic
Child:    Son, Daughter, Stepson, or Stepdaughter of Passenger
           Aboard Titanic

Other family relatives excluded from this study include cousins,
nephews/nieces, aunts/uncles, and in-laws.  Some children travelled
only with a nanny, therefore parch=0 for them.  As well, some
travelled with very close friends or neighbors in a village, however,
the definitions do not support such relations.
```

The Titanic dataset

```
full_titanic %>%
  group_by(Survived,
           Sex) %>%
  count()
```

```
## # A tibble: 4 x 3
## # Groups:   Survived, Sex [4]
##   Survived Sex        n
##      <dbl> <chr>  <int>
## 1        0 female    81
## 2        0 male     468
## 3        1 female   233
## 4        1 male     109
```

```
full_titanic %>%
  group_by(Sex) %>%
  summarise(p = mean(Survived),
            Y = sum(Survived),
            N = n())
```

```
## # A tibble: 2 x 4
##   Sex        p     Y     N
##   <chr>  <dbl> <dbl> <int>
## 1 female 0.742   233   314
## 2 male   0.189   109   577
```

The Titanic dataset

```
##
## Call:
## glm(formula = Survived ~ Age + Pclass, family = binomial(), data = full_titanic)
##
## Deviance Residuals:
##     Min       1Q   Median       3Q      Max
## -2.1524  -0.8466  -0.6083   1.0031   2.3929
##
## Coefficients:
##              Estimate Std. Error z value Pr(>|z|)
## (Intercept)  2.296012   0.317629   7.229 4.88e-13 ***
## Age         -0.041755   0.006736  -6.198 5.70e-10 ***
## Pclass2     -1.137533   0.237578  -4.788 1.68e-06 ***
## Pclass3     -2.469561   0.240182 -10.282  < 2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## (Dispersion parameter for binomial family taken to be 1)
##
##     Null deviance: 964.52  on 713  degrees of freedom
## Residual deviance: 827.16  on 710  degrees of freedom
##   (177 observations deleted due to missingness)
```

The Titanic dataset

```r
library(emmeans)
emmeans(age_class,
        ~Age|Pclass,
        type = "response")
```

```
## Pclass = 1:
##   Age  prob     SE  df asymp.LCL asymp.UCL
##  29.7 0.742 0.0339 Inf     0.670     0.803
##
## Pclass = 2:
##   Age  prob     SE  df asymp.LCL asymp.UCL
##  29.7 0.480 0.0394 Inf     0.403     0.557
##
## Pclass = 3:
##   Age  prob     SE  df asymp.LCL asymp.UCL
##  29.7 0.196 0.0216 Inf     0.157     0.241
##
## Confidence level used: 0.95
## Intervals are back-transformed from the logit scale
```

Some final notes on Generalized Linear Models

Today has focussed on **logistic** regression with *binomial* distributions.

But Generalized Linear Models can be expanded to deal with many different types of outcome variable!

e.g. *Counts* follow a Poisson distribution - use `family = "poisson"`

Ordinal variables (e.g. Likert scale) can be modelled using *cumulative logit* models (using the **ordinal** or **brms** packages).

Suggested reading for categorical ordinal regression

Liddell & Kruschke (2018). Analyzing ordinal data with metric models: What could possibly go Wrong?

Buerkner & Vuorre (2018). Ordinal Regression Models in Psychology: A Tutorial